MODAL TRANSLATIONS AND INTUITIONISTIC DOUBLE NEGATION

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Abstract

The purpose of this paper is to show a connection between the embedding of Heyting's logic in S4 with the translation which prefixes the necessity operator $\Box$ to every subformula and the embedding of classical logic in Heyting's logic with the translation which prefixes double negation $\neg\neg$ to every subformula. This is done by introducing some intuitionistic modal logics.

With the first translation, called t, Heyting's first-order logic is embedded in a modal system of the S4 type based on classical first-order logic (its modal postulates are: necessitation, $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$, $\neg\Box\neg(A \rightarrow A)$ and $\Box A \leftrightarrow \Box\Box A$; $\Box A \rightarrow A$ is lacking). Again with t, classical first-order logic is embedded in a modal system of the S5 type based on Heyting's first-order logic (in addition to the modal postulates above it has $\Box(\Box A \vee \Box\neg\Box A)$). If now we introduce in Heyting's logic an operator $\Box$ and assume $\Box A \leftrightarrow \neg\neg A$, the translation with $\neg\neg$ becomes an instance of t, and the resulting modal system is an extension of the modal system of the S5 type mentioned above. As in S4 we assume about $\Box$ more than strictly needed for embedding Heyting's logic ($\Box A \rightarrow A$ is superfluous), so in Heyting's logic we assume about $\neg\neg$ more than strictly needed for embedding classical logic ($A \rightarrow \neg\neg A$ is superfluous). Minimal normal modal systems needed for these embeddings in the propositional case are even slightly weaker than the systems above (they have $\neg\Box\neg(A \rightarrow A)$ and $\Box A \leftrightarrow \Box\Box A$ with a $\Box$ prefixed). Finally, Kripke-style models for the propositional calculi corresponding to the modal systems introduced are considered. Correspondence results for various modal axioms, and soundness and completeness results are stated. Results for intuitionistic propositional modal systems follow easily from some previous work done on this topic (which has not yet been extended to cover predicate calculi; also a certain difficulty involving the Barcan formula arises for the Heyting predicate calculus with $\Box A \leftrightarrow \neg\neg A$).

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Introduction

There are two famous kinds of translations connected with Heyting's logic. The first kind, which stems from [10] and [15], is based on prefixing the necessity operator $\Box$ to subformulae of a nonmodal formula. With translations of this kind Heyting's logic can be embedded in the modal system S4. The second kind of translation, which stems from [12], [11] and [9], is based on prefixing double negation $\neg\neg$ to subformulae. With translations of this kind classical logic can be embedded in Heyting's logic. Often in these translations $\Box$ and $\neg\neg$ are not prefixed to certain subformulae, and in some cases prefixed subformulae can be replaced by various equivalents. However, in this paper we shall concentrate on the two translations which simply prefix $\Box$ and $\neg\neg$ to *every* subformula (cf. [16], [7] and [12], [13], [3] p. 208, 38.12, [14] pp. 41ff). We shall try to show that the embeddings based on these two translations are more similar than it seems to be usually realized. For that purpose we shall introduce some intuitionistic modal logics.

Using the translation with $\Box$ we shall embed Heyting's first-order logic into a modal system of the S4 type based on classical first-order logic. With this translation we shall also embed classical first-order logic into a modal system of the S5 type based on Heyting's first-order logic. If now we introduce in Heyting's logic an operator $\Box$ and assume $\Box A \leftrightarrow \neg\neg A$, the translation with $\neg\neg$ becomes an instance of the translation with $\Box$, and the resulting modal system is an extension of the modal system of the S5 type based on Heyting's logic which we have just mentioned.

In § 1 and § 2 we introduce our first-order modal systems and we present our embedding results. In § 3 we consider the question what are the minimal properties we must assume for $\Box$ in order to be able to prove our embeddings. In § 4 we consider Kripke-style models for the propositional calculi corresponding to our modal systems. We shall give correspondence results for our modal axioms, and also soundness and completeness results. We shall concentrate on the propositional calculi only because we can easily deduce results in this area from what is known about modal logic with a classical or intuitionistic base. Models for intuitionistic predicate modal logic seem to be rather unexplored, and we would need some preparatory work on this topic before tackling particular systems like those we shall introduce. (At the very end of our paper we shall hint at a particular difficulty involving the Barcan formula which seems to arise for the models of the Heyting predicate calculus with $\Box A \leftrightarrow \neg\neg A$.)

§ 1 *First-Order Modal Logics*

Let L be the language of the first-order predicate calculus which has denumerably many individual variables $x, x_1, x_2, \ldots$, denumerably many individual constants, denumerably many predicate constants, and the logical constants $\rightarrow$, $\wedge$, $\vee$, $\neg$, $\forall$ and $\exists$. As schematic letters for formulae we shall use A, B, C, D, $A_1, \ldots$ As usual, $A \leftrightarrow B$ abbreviates $(A \rightarrow B) \wedge (B \rightarrow A)$. The schema A(x) stands for a formula in which eventually x occurs free, and A(a) is obtained from A(x) by substituting the individual variable or individual constant a for free x, with the usual provisos for substitution.

The Heyting first-order predicate calculus H is axiomatized in L with the following usual axiom-schemata and rules:

1. $A \rightarrow (B \rightarrow A)$, 2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
3. $(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow (A \wedge B)))$, 4. $(A \wedge B) \rightarrow A$,
5. $(A \wedge B) \rightarrow B$,
6. $A \rightarrow (A \vee B)$, 7. $B \rightarrow (A \vee B)$, 8. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$,
9. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$, 10. $\neg A \rightarrow (A \rightarrow B)$,

11. $\dfrac{A \qquad A \rightarrow B}{B}$,

12. $\forall x A(x) \rightarrow A(a)$, 13. $A(a) \rightarrow \exists x A(x)$,

14. $\dfrac{B \rightarrow A}{B \rightarrow \forall x A}$, 15. $\dfrac{A \rightarrow B}{\exists x A \rightarrow B}$,

provided x does not occur free in B in the last two rules.

The classical first-order predicate calculus C is obtained by extending H with $A \vee \neg A$.

Now let $L_\Box$ be L extended with the necessity operator $\Box$. The system in $L_\Box$ which we shall call H4 is axiomatized by extending H with the following axiom-schemata and rules:

$\Box 1.$ $\Box(A \to B) \to (\Box A \to \Box B)$, $\qquad$ $\Box 2.$ $\dfrac{A}{\Box A}$,

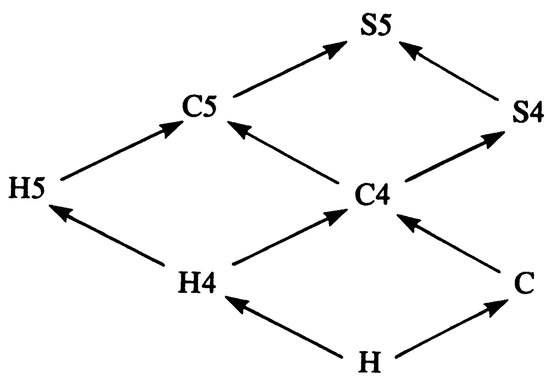
$\Box 3.$ $\neg\Box\neg(A \to A)$, $\qquad$ $\Box 4.$ $\Box A \leftrightarrow \Box\Box A$.

The system H5 is axiomatized by extending H4 with

$$\Box 5. \ \Box(\Box A \vee \Box\neg\Box A).$$

The systems C4 and C5 are obtained by extending H4 and H5 respectively with $A \vee \neg A$. The first-order modal calculi S4 and S5 are obtained by extending C4 and C5 respectively with $\Box A \to A$.

It is not difficult to show that the following relations of proper inclusion hold between the systems we have now introduced (arrows indicate proper inclusion):



These relations can be deduced either from the difference between systems in L and systems in $L_\Box$, or from the proper containment of S4 in S5, or from the proper containment of H in C, or from the lack of $\Box A \to A$ in systems below S4 and S5 (this last fact will follow easily from the results of § 4).

§ 2 *Embedding Heyting's and Classical First-Order Logic in Modal Logics*

We shall say that a system S_1 can be embedded in a system S_2 by a translation (1-1 mapping) m from the language of S_1 into the language of S_2 iff $\vdash_{S_1} A \Leftrightarrow \vdash_{S_2} m(A)$. Let t(A) be the result of prefixing □ to every subformula of the formula A of L. More precisely, t(A) is defined recursively as follows:

$$\begin{array}{lll} t(A) & = \Box A & \text{if } A \text{ is atomic} \\ t(A\,\alpha\,B) & = \Box(t(A)\,\alpha\,t(B)) & \text{if } \alpha \text{ is } \rightarrow, \wedge, \text{ or } \vee \\ t(\beta A) & = \Box\beta\, t(A) & \text{if } \beta \text{ is } \neg, \forall x, \text{ or } \exists x. \end{array}$$

Then we can prove the following theorems (where we assume that S_1 and S_2 are included among the systems *between* S_1 and S_2):

Theorem 1. H can be embedded by t in every system between H4 and S4.

Theorem 2. C can be embedded by t in every system between H5 and S5.

Proofs. 1. To show that $\vdash_H A \Rightarrow \vdash_{H4} t(A)$ we make a straightforward induction on the length of proof of A in H. For the converse we use the fact that H4 is a subsystem of S4, and it is well-known that H can be embedded in S4 by t (see [16] and [7]). Since H can be embedded in both H4 and S4, it can be embedded in all the systems in between.
2. To show that $\vdash_C A \Rightarrow \vdash_{H5} t(A)$ we again make a straightforward induction on the length of proof of A in C. For the converse suppose $\vdash_{H5} t(A)$ and let Ctriv be the system in $L_\Box$ obtained by extending C with $\Box A \leftrightarrow A$. Then, since H5 is contained in Ctriv, we have $\vdash_{Ctriv} t(A)$, which yields $\vdash_{Ctriv} A$; and since Ctriv is a conservative extension of C in L, we have $\vdash_C A$. So, C can be embedded in both H5 and Ctriv, which implies that it can be embedded in all the systems in between, and in particular in all the systems between H5 and S5. q.e.d.

An embedding of a nonmodal logic S_1 into a modal logic S_2 is more interesting when the underlying nonmodal logic of S_2 is different from S_1. So we single out the following two consequences of our theorems:

H can be embedded by t in C4,
C can be embedded by t in H5.

In other words, H can be interpreted from the classical standpoint if we introduce a □ of the S4 type like the one in C4, and C can be interpreted from the intuitionistic standpoint if we introduce a □ of the S5 type like the one in H5.

Whereas in C we apparently cannot define a □ of the S4 type sufficient for our embedding (in the propositional case this impossibility follows immediately from the nonexistence of a truth table decision procedure for the Heyting propositional calculus), in H we already have a □ of the S5 type sufficient for our embedding, though somewhat stronger than the □ of H5. This operator is $\neg\neg$.

Let Hdn be the system in $L_\Box$ obtained by extending H with

□dn. $\Box A \leftrightarrow \neg\neg A$.

It is easy to show that H5 is contained in Hdn. The converse doesn't hold (note that $A \rightarrow \Box A$ can't be provable in H5; see also § 4). It is also easy to show that Hdn and C5 (or S5) are not contained in each other. Finally, it is not difficult to conclude that C can be embedded by t in Hdn, since this embedding now boils down to the embedding of C in H by the translation which prefixes $\neg\neg$ to every subformula (see [3] p. 208, 38.12, [16] and [14] pp. 41ff). All this shows that $\neg\neg$ can be conceived as a □ operator having some properties in addition to those assumed for H5, which however don't spoil the possibility of embedding C by t. This is analogous to the fact that the S4 operator □ has some properties in addition to those assumed for C4, which however don't spoil the possibility of embedding H by t.

In what respect Hdn is stronger than H5 can be realized from an alternative axiomatization of Hdn. This system can be obtained by extending H5 with

□dn1. $A \rightarrow \Box A$
□dn2. $\Box(((A \rightarrow B) \rightarrow A) \rightarrow A)$

(□2, □4 and □5 are superfluous in this axiomatization). It is clear that both □dn1 and □dn2, as well as the modal postulates of H5, hold in the presence of □dn. Below we sketch the proof of □dn in this extension of H5 (besides the postulates marked we also use □1 and □2, the latter being a consequence of □dn1):

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\Box\neg(A\to A)\quad \Box 3}{\neg\Box(A\wedge\neg A)}}{\neg(\Box A\wedge\Box\neg A)}}{\Box A\to\neg\Box\neg A}\qquad \dfrac{\neg A\to\Box\neg A\quad \Box}{\neg\Box\neg A\to\neg\neg A}}{\Box A\to\neg\neg A;}$$

$$\dfrac{\Box\text{dn1}\quad \neg\neg A\to\Box\neg\neg A\qquad \dfrac{\dfrac{\Box(((A\to(B\wedge\neg B))\to A)\to A)\quad \Box}{\Box(\neg\neg A\to A)}}{\Box\neg\neg A\to\Box A}}{\neg\neg A\to\Box A.}$$

Now, □dn2 is provable in C4, and of course we could embed C by t in H5 extended with □dn2. Hence, the significant new modal postulate added by Hdn when extending H5 is □dn1, i.e. A→□A, wich is the exact converse of □A→A, the significant new modal postulate added by S4 when extending C4. This stresses the analogy between the □ of Hdn and the □ of S4 we have noted above.

§ 3 *Minimal Modal Logics in which Heyting's and Classical Logic can be Embedded*

The remarks at the end of the last section lead us to the question: "What are the minimal properties we must assume for □ in order to be able to embed H or C by t?" Of course, the minimal modal systems in which we can embed H or C by t are those which contain just the t-translations of the theorems of H or C. Our question might receive a less trivial answer if we require from our minimal modal systems that they be *normal*, where following a terminology common in modal logic we shall say that a modal system containing either H or C (or the corresponding propositional calculi) is normal iff □1 is provable in it and it is closed under □2. All the modal systems we have considered are normal. However, it is possible that even in H4 (or C4) and H5 (or C5) we assume about □ somewhat more than strictly required from a normal modal system in which we can embed H and C respectively by t.

Let us consider this question of minimal normal modal systems in the propositional case. Our propositional modal systems will be formulated in a propositional language $Lp_\Box$ with the connectives $\to$, $\wedge$, $\vee$, $\neg$ and $\Box$. The system Hp is the Heyting propositional calculus axiomatized with 1-11, and Cp is the classical propositional calculus obtained by extending Hp with $A \vee \neg A$. The systems H4p, H5p, Hdnp, C4p, C5p, S4p and S5p are obtained by extending Hp or Cp with the corresponding axiom-schemata and rules analogously to what we had in § 1 and § 2. All these modal systems are normal.

It is easy to see that exact analogues of Theorems 1 and 2 can be proved if we restrict ourselves to the propositional calculi we have just introduced.

Let now $C4p^-$ and $H5p^-$ be the systems obtained from C4p and H5p respectively by replacing $\Box 3$ and $\Box 4$ by $\Box\neg\Box\neg(A \to A)$ and $\Box(\Box A \leftrightarrow \Box\Box A)$ (these two schemata are $\Box 3$ and $\Box 4$ with a $\Box$ prefixed; since with $\Box 1$ and $\Box 2$ we have $(\Box B \wedge \Box C) \leftrightarrow \Box(B \wedge C)$, the schema $\Box(\Box A \leftrightarrow \Box\Box A)$ can be replaced by $\Box(\Box A \to \Box\Box A)$ and $\Box(\Box\Box A \to \Box A)$). The systems $C4p^-$ and $H5p^-$ are proper sub-systems of C4p and H5p respectively (as will become apparent in § 4).

We can prove the following theorems:

Theorem 3 (see [4] Theorem 5.1). $C4p^-$ is the minimal normal propositional modal system containing Cp in which Hp can be embedded by t.

Theorem 4. $H5p^-$ is the minimal normal propositional modal system containing Hp in which Cp can be embedded by t.

Proofs. By inductions on the length of proof of A we show that $\vdash_{Hp} A \Rightarrow \vdash_{C4p^-} t(a)$ and $\vdash_{Cp} A \Rightarrow \vdash_{H5p^-} t(a)$. In these inductions we use the fact that $C4p^-$ and $H5p^-$ are closed under the rule of replacement

$$\Box\text{rep.} \quad \frac{\Box(A \leftrightarrow B) \qquad \Box D}{\Box D\,[A/B]}$$

where D [A/B] is the result of substituting zero or more occurrences of A in D by B. These closures are proved by induction on the complexity of D. The converse implications $\vdash_{C4p^-} t(a) \Rightarrow \vdash_{Hp} A$ and

$\vdash_{H5p^-} t(A) \Rightarrow \vdash_{\overline{C}p} A$ follow from the containment of C4p$^-$ and H5p$^-$ in C4p and H5p respectively.

The minimality of C4p$^-$ follows from the fact that $\Box\neg\Box\neg\Box(\Box A \rightarrow \Box A)$, $\Box(\Box A \rightarrow \Box(\Box(\Box B \rightarrow \Box B) \rightarrow \Box A))$ and $\Box(\Box(\Box(\Box B \rightarrow \Box B) \rightarrow \Box A) \rightarrow \Box A)$ are t-translations of theorems of Hp. For the minimality of H5p$^-$ we have in addition $\Box 5$ which is a t-translation of $A \vee \neg A$. q.e.d.

However, it seems these results cannot be straightforwardly extended to the corresponding predicate calculi C4$^-$ and H5$^-$ (in spite of what a remark at the end of [4] might suggest). Proofs of theorems which would be analogues of Theorems 3 and 4 run into difficulties in showing that C4$^-$ and H5$^-$ are closed under the rule $\Box$rep mentioned in the proofs above. (It is not clear how to pass from $\vdash \Box(A \leftrightarrow B)$ and $\vdash \Box \forall x A$ to $\vdash \Box \forall x B$, and analogously with $\exists$.) So, we shall leave open the question: "What are the minimal normal first-order modal systems containing C or H in which the predicate calculi H or C can be embedded by t?".

This question is simplified a little bit if we consider normal first-order modal systems with the Barcan formula (BF) $\forall x \Box A \rightarrow \Box \forall x A$. Let C4$^-$ + BF and H5$^-$ + BF be the first-order systems corresponding to C4p$^-$ and C5p$^-$, extended with BF. Then we can easily prove by inductions on the length of proof of A that $\vdash_{\overline{H}} A \Rightarrow \vdash_{\overline{C4}^- + BF} t(A)$ and $\vdash_{\overline{C}} A \Rightarrow \vdash_{H5^- + BF} t(A)$. (Closure under $\Box$rep for C4$^-$ + BF and H5$^-$ + BF is now readily shown.) The converse of the last implication is easily obtained using Ctriv as in the proof of Theorem 2. (Note that a similar move is not available for the converse of the other implication since C4$^-$ + BF is not a subsystem of H extended with $\Box A \leftrightarrow A$.) Hence, we can conclude that H5$^-$ + BF is the minimal normal first-order modal system with BF containing H in which C can be embedded by t, and we conjecture that an analogous result could be obtained for C4$^-$ + BF and the embedding of H. Instead of BF we could have used throughout something which follows from adding BF, i.e. closure under the rule

$$\frac{\forall x \Box A}{\Box \forall x A.}$$

We have not considered systems with BF till now (save for S5, in which BF is provable) because in at least one of our modal systems, namely Hdn, this formula is undesirable. In Hdn with BF we would obtain $\forall x \neg\neg A \rightarrow \neg\neg \forall x A$, which is unprovable in H (cf. also the end of § 4). However, adding BF to Hdn (which here does not differ from adding the rule derived from BF which we have mentioned above) does not spoil the possibility of embedding C by t.

To conclude, it seems safe to say that even if C4 and H5 are not exactly the minimal normal modal systems we need for our embeddings, they are not much stronger than these minimal systems.

§ 4 *Models for Propositional Modal Logics*

Now we shall consider Kripke-style models for the modal propositional calculi we have introduced in § 3. For the systems based on Cp these shall be the usual kind of Kripke models for propositional modal logic, whereas for the systems based on Hp we shall have models with two "accessibility relations", one intuitionistic and the other modal. This latter kind of models was investigated extensively in [1] and [5].

First we summarize some terminology and results of [1]. An H$\Box$ frame is $<X, R_I, R_M>$ where $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$ and $R_I R_M \subseteq R_M R_I$ ($R_1 R_2$ is short for $R_1 \circ R_2$). The variables $u, v, w, u_1, \ldots$ range over X. An H$\Box$ model is $<X, R_I, R_M, V>$ where $<X, R_I, R_M>$ is an H$\Box$ frame and the valuation V is a mapping from the set of propositional variables of $Lp_\Box$ to the power set of X such that the following heredity condition is satisfied for every propositional variable p: $\forall u, v(u R_I v \Rightarrow (u \in V(p) \Rightarrow v \in V(p)))$. The relation $\models$ in $u \models A$ is defined as usual, except that for $\rightarrow$ and $\neg$ it involves R_I as in intuitionistic Kripke models, whereas for the necessity operator $\Box$ it involves R_M as in modal Kripke models. A formula A holds in a model $<X, R_I, R_M, V>$ iff $(\forall u \in X)\ u \models A$; A holds in a frame Fr (i.e. $Fr \models A$) iff A holds in every model with this frame; and A is valid iff A holds in every frame. An H$\Box$ frame (model) is condensed iff $R_I R_M = R_M$, and it is strictly condensed iff $R_I R_M = R_M R_I = R_M$. The system HKp, i.e. Hp extended with $\Box$1 and $\Box$2, is sound and complete with respect to H$\Box$ frames (condensed H$\Box$ frames, strictly condensed H$\Box$ frames).

A C$\Box$ frame $<X, R_M>$ is a standard Kripke frame for propositional modal logic, and a C$\Box$ model $<X, R_M, V>$ is a corresponding model, where V has no heredity condition to satisfy. The definition of $\models$ and other related definitions are also standard (see, for example, [2]).

We shall now state what conditions on H$\Box$ frames and C$\Box$ frames correspond to schemata characteristic for our modal systems. The proofs of the equivalences stating this correspondence are quite analogous to proofs which can be found in [5], and we shall omit them. From now on, $R_\Box$ will be an abbreviation for $R_M R_I$.

Let Fr be an H$\Box$ frame. Then

(1) $Fr \models \neg\Box\neg(A \to A)$ $\Leftrightarrow \forall v\, \exists w\, vR_\Box w$ (i.e. $R_\Box$ is *serial*)
(1a) $Fr \models \Box\neg\Box\neg(A \to A)$ $\Leftrightarrow \forall u, v\, \exists w(uR_\Box v \Rightarrow vR_\Box w)$
(2) $Fr \models \Box A \to \Box\Box A$ $\Leftrightarrow R_\Box{}^2 \subseteq R_\Box$ (i.e. $R_\Box$ is *transitive*)
(2a) $Fr \models \Box(\Box A \to \Box\Box A)$ $\Leftrightarrow \forall u, v, w(uR_\Box v \Rightarrow (vR_\Box{}^2 w \Rightarrow vR_\Box w))$
(3) $Fr \models \Box\Box A \to \Box A$ $\Leftrightarrow R_\Box \subseteq R_\Box{}^2$ (i.e. $R_\Box$ is *weakly dense*)
(3a) $Fr \models \Box(\Box\Box A \to \Box A)$ $\Leftrightarrow \forall u, v, w(uR_\Box v \Rightarrow (vR_\Box w \Rightarrow vR_\Box{}^2 w))$
(4) $Fr \models \Box A \vee \Box\neg\Box A$ $\Leftrightarrow R_\Box{}^{-1} R_\Box \subseteq R_\Box$ (i.e. $R_\Box$ is *euclidean*)
(4a) $Fr \models \Box(\Box A \vee \Box\neg\Box A)$ $\Leftrightarrow \forall u, v, w(uR_\Box v \Rightarrow (vR_\Box{}^{-1} R_\Box w \Rightarrow vR_\Box w))$.

The conditions on the right-hand sides of (1)-(4) are modified on the right-hand sides of (1a)-(4a) by adding an assumption. The resulting conditions are called *conditional seriality, conditional transitivity*, etc. It is easily inferred from (2) and (3), and (2a) and (3a), that the conditions corresponding to $\Box A \leftrightarrow \Box\Box A$ and $\Box(\Box A \leftrightarrow \Box\Box A)$ are respectively $R_\Box = R_\Box{}^2$ and its conditional variant (remember that in HKp we can prove $(\Box B \wedge \Box C) \leftrightarrow \Box(B \wedge C)$).

If Fr is a strictly condensed H$\Box$ frame or a C$\Box$ frame, then all our equivalences hold when we substitute R_M for $R_\Box$.

Let us also consider the correspondence between the schemata characteristic for Hdnp and conditions on H$\Box$ frames (see [6]):

(5) $Fr \models A \to \Box A$ $\Leftrightarrow R_\Box \subseteq R_I$
(6) $Fr \models \Box(((A \to B) \to A) \to A)$ $\Leftrightarrow \forall u, v, w(uR_\Box v \Rightarrow (vR_I w \Rightarrow wR_I v))$.

Using our equivalences and fairly standard proofs with canonical models, of the type to be found in [1], [5] and [6], we can show the soundness and completeness of the systems named on the left with respect to H$\Box$ frames which satisfy the conditions mentioned in the equivalences referred to on the right:

H4p	: (1), (2), (3)
H5p	: (1), (2), (3), (4a)
Hdnp	: (1), (5), (6)
H5p$^-$	: (1a), (2a), (3a), (4a).

These soundness and completeness results still hold if we restrict ourselves to condensed or strictly condensed H$\Box$ frames.

It is a matter of routine to show also that systems based on Cp named on the left are sound and complete with respect to C$\Box$ frames which satisfy the conditions mentioned in the equivalences referred to on the right, provided we substitute R_M for $R_\Box$ in these conditions:

C4p	: (1), (2), (3)
C4p$^-$	: (1a), (2a), (3a)
C5p	: (1), (2), (3), (4a)
S4p	: (2), R_M is reflexive
S5p	: (2), (4), R_M is reflexive.

Having all this in mind it is not difficult to construct models of modal predicate logic which will serve to demonstrate the various proper containement relations we have claimed for the modal predicate calculi of this paper.

We have said above that soundness and completeness for systems based on Hp hold with strictly condensed H$\Box$ frames. Strictly condensed H$\Box$ frames with respect to which Hdnp can be shown sound and complete are especially interesting. If in these frames R_I is a partial ordering, our soundness and completeness result still holds. These frames can be characterized more simply as partially ordered frames where for any u there is a maximal element v above u, uR_Mv means that v is a maximal element above u, and $u \models \Box A$ means that A holds in all maximal elements above u (see [6]). So these frames are analogous to the frames with respect to which the Heyting predicate calculus with the Double Negation Shift (DNS) formula $\forall x \neg\neg A \rightarrow \neg\neg \forall x A$ should be sound and complete (see [8] pp. 41, 57-58). The DNS formula becomes equivalent to the Barcan formula $\forall x \Box A \rightarrow \Box \forall x A$ in the presence of $\Box A \leftrightarrow \neg\neg A$. Since in H we cannot prove DNS, in Hdn we should not be able to prove DNS or the Barcan formula. So it seems there might be difficulties in transforming the frames for Hdnp into frames for the predicate calculus Hdn. These difficulties appear not only with our strictly condensed partially ordered frames, but also with first-order variants of ordinary H$\Box$ frames adequate for Hdn, which seem to validate $\Box \forall x(A \vee \neg A)$. (The schema $\neg\neg \forall x(A \vee \neg A)$ can replace DNS in H extended with DNS.) However, we leave for another occasion the topic of models for the intuitionistic first-order modal logics we have introduced.

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